

## Sums of Distinct Units

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**Summary.** It is shown that  $Q(2^{1/2})$  and  $Q(5^{1/2})$  are the only quadratic fields in which every integer is a sum of distinct units. This confirms a conjecture of Jacobson [2].

1. Denote by  $DU$  the class of all algebraic number fields  $K$  with the following property:

*Every integer of  $K$  is a sum of distinct units of  $K$ .*

Jacobson [2] has shown that  $Q(\sqrt{2}) \in DU$  and indicated that the same method can be used to prove  $Q(\sqrt{5}) \in DU$ . He conjectured that no other quadratic field belongs to  $DU$ , and noted that

$$Q(\sqrt{3}), Q(\sqrt{6}), Q(\sqrt{7}) \notin DU.$$

In this note we confirm Jacobson's conjecture.

2. **THEOREM.** *A quadratic field  $K=Q(\sqrt{d})$ , ( $d$  — squarefree) belongs to  $DU$  if and only if  $d=2$  or  $d=5$ .*

**Proof.** We need an easy lemma:

**LEMMA.** *Assume that the polynomial*

$$\varphi(x) = A_n x^n + \dots + A_0 \in Z[x] \quad (A_n \neq 0)$$

*has a root  $x_0$  with  $|x_0| > 2$ . Let  $h = \max_{0 \leq i < n} |A_i|$ ,  $h > 1$ , and let  $|A_k| = h$ . If for  $i \neq k$   $|A_i| \leq 1$ , then*

$$|x_0|^{n-k} < \frac{(h-1)(|x_0|^n - 1)}{|x_0| - 2}.$$

**Proof of the lemma.** Obviously  $k \neq n$ . As

$$|A_k x_0^k| \leq h |x_0|^k$$

and

$$\sum_{j \neq k, n} |A_j x_0^j| \leq \frac{|x_0|^n - |x_0|^{k+1} + |x_0|^k - 1}{|x_0| - 1}$$

we get

$$|x_0|^n (|x_0| - 2) \leq |x_0|^{nk} (h-1) (|x_0| - 1) - 1$$

which easily implies our assertion.

Assume now that  $Q(\sqrt{d}) \in DU$ . Obviously  $d > 0$ . Write

$$(1) \quad 2 = \pm \varepsilon^{k_1} \pm \varepsilon^{k_2} \pm \dots \pm \varepsilon^{k_s},$$

where  $\varepsilon = a + b\theta$  is that fundamental unit of  $K$  which satisfies  $a \geq 0$ ,  $b > 0$ . Here  $\theta$  is either  $\sqrt{d}$  (if  $d \not\equiv 1 \pmod{4}$ ) or  $(1 + \sqrt{d})/2$  ( $d \equiv 1 \pmod{4}$ ).

We can write (1) in the form

$$A_n \varepsilon^n + \dots + A_0 = 0,$$

where  $A_n, A_0 \neq 0$ ,  $A_i \in \mathbb{Z}$ , and at most one of the  $A_i$  equals  $\pm 3$  or  $\pm 2$ , whereas all others equal  $\pm 1$  or 0. One checks immediately that  $|\varepsilon| < 4$ .

If  $d \not\equiv 1 \pmod{4}$ , then this implies

$$\sqrt{d} \leq |a + b\sqrt{d}| = |\varepsilon| < 4,$$

thus  $d < 16$ . If however  $d \equiv 1 \pmod{4}$ , and  $a \geq 1$  then

$$1 + \frac{1 + \sqrt{d}}{2} \leq \varepsilon < 4, \quad \text{so } d < 25.$$

Finally, if  $d \equiv 1 \pmod{4}$  and  $a = 0$ , then necessarily  $d = 5$ .

Now  $|\varepsilon| < 4$  hold in our range only for  $d = 2, 3, 5, 13$ . So let us look at  $Q(\sqrt{3})$ . Here  $\varepsilon = (3 + \sqrt{13})/2 > 2$ . Write

$$\sqrt{5} = \pm \varepsilon^{k_1} \pm \varepsilon^{k_2} \pm \dots \pm \varepsilon^{k_s},$$

this implies an equality of the form  $\varphi(x) = 0$ , where

$$\varphi(x) = A_n x^n + \dots + A_0$$

with one of the coefficients, say  $A_k$ , equal to  $\pm 4$ ,  $\pm 5$  or  $\pm 6$  and the remaining equal to  $\pm 1$  or 0. Applying the lemma we get  $n - k \leq 2$ . But  $\varphi(-\varepsilon^{-1}) = 0$  because  $\varepsilon$  and  $-\varepsilon^{-1}$  are conjugated, thus  $-\varepsilon$  is a root of

$$A_0 x^n + \dots + A_n = 0$$

and again by the lemma one gets  $k \leq 2$ . Thus  $n \leq 4$ , but the minimal polynomial  $f(x)$  of  $\varepsilon$  equals  $x^2 - 3x - 1$  and an easy check shows that it cannot divide any polynomial of degree  $\leq 4$  with all coefficients except one equal to  $\pm 1$  or 0, with the exceptional coefficient being equal  $\pm 4$ ,  $\pm 5$ ,  $\pm 6$ .

The same argument enables us to get rid of the case  $d = 3$ .

Because the inclusions  $Q(\sqrt{2})$ ,  $Q(\sqrt{5}) \in DU$  are proved in [2] the theorem is proved.

3. Observe that the same method can be utilized to show that no pure cubic field  $Q(\sqrt[3]{m})$  is in  $DU$ . One has only to utilize a result of Delaunay [1] who proved that in  $Q(\sqrt[3]{m})$  one can find a fundamental unit of the form  $a + b\sqrt[3]{m} + c\sqrt[3]{m}^2$  with  $a, b, c \in Z$  and positive.

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#### REFERENCES

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[2] B. Jacobson, *Sums of distinct divisors and sums of distinct units*, Proc. Amer. Math. Soc., **15** (1964), 179—183.

Я. Слива, Суммы различных единиц

Содержание. В представленной работе доказано, что  $Q(2^{1/2})$  и  $Q(5^{1/2})$  — единственные квадратичные поля, в которых всякое целое число является суммой различных единиц. Вышеприведенное подтверждает предположение Джекобсона.