

On the Length of Factorizations in Algebraic Number Fields

by
 J. ŚLIWA

Presented by K. URBANIK on December 10, 1974

Summary. In this note we announce some generalizations of Carlitz's theorem and also some asymptotic formulas for the number of nonassociated integers α of a field K whose factorizations into irreducibles in K have the same number of lengths, and $|N(\alpha)| \leq x$.

1. Let K be a finite extension of the rationals, R_K — the ring of integers of K , $H(K)$ — the class-group of the field K , $I(K)$ — the group of ideals, P — the set of all prime ideals and finally let $f: I(K) \rightarrow H(K)$ be the natural homomorphism.

We will write $G(K)$ for the set of all integers whose all factorizations into irreducibles in R_K have the same length and $G'(K)$ for $G(K) \cap \mathbb{N}$, \mathbb{N} — the set of natural numbers.

Now we introduce some definitions:

DEFINITION 1. The sets A_1, A_2, \dots, A_n will be called a partition of R_K if:

1. A_1, A_2, \dots, A_n are multiplicative subsets of R_K .
2. If $a \in A_i$, $b|a$ then $b \in A_i$, for $i=1, 2, \dots, n$.
3. There exists $m > 0$ such that $A_1 A_2 \dots A_n$ contains all m -th powers of R_K .

DEFINITION 2. The partition A_1, A_2, \dots, A_n will be called a good partition if $A_i \subset G(K)$ for $i=1, 2, \dots, n$.

If G is a finite abelian group, $g_1, g_2, \dots, g_k \in G$ then the equality

$$(1) \quad g_1^{n_1} g_2^{n_2} \dots g_k^{n_k} = 1$$

will be called a minimal equality if $0 < n_i \leq \text{ord } g_i$ for $i=1, \dots, k$ and if $0 \leq m_i \leq n_i$, $i=1, 2, \dots, k$,

$$g_1^{m_1} g_2^{m_2} \dots g_k^{m_k} = 1$$

implies $(m_1, m_2, \dots, m_k) = (n_1, n_2, \dots, n_k)$ or $(0, 0, \dots, 0)$. The minimal equality (1) will be said to satisfy condition (*) if

$$\sum_{i=1}^k \frac{n_i}{\text{ord } g_i} = 1.$$

Besides, if U is any subset of the group G , then U will be said to satisfy (*) (and we will write $U \in (*)$) if every minimal equality between the elements of U satisfies condition (*).

2. If for any subset H_1 of the class-group $H(K)$ we denote

$$I(H_1) = f^{-1}(H_1) \cap P,$$

$$R_K(H_1) = \{a \in R_K : \mathfrak{P} | aR_K, \mathfrak{P} \in P \text{ implies } \mathfrak{P} \in I(H_1)\}.$$

Then we have

THEOREM 1. $R_K(H_1) \subset G(K)$ if and only if $H_1 \in (*)$.

COROLLARY. The partition A_1, A_2, \dots, A_n is a good partition if and only if $H_i \in (*)$, $i = 1, 2, \dots, n$, where

$$H_i = f(\{\mathfrak{P} \in P : \text{there exists } a \in A_i, \mathfrak{P} | aR_K\}).$$

We write $l(K)$ for the minimal number n such that there exists a good partition of R_K consisting of n sets. If

$$H(K) = \{1, X_1, \dots, X_{h-1}\}$$

then the sets

$$H_1 = \{1, X_1\}, \quad H_2 = \{X_2\}, \dots, \quad H_{h-1} = \{X_{h-1}\}$$

satisfy condition (*) and therefore $l(K) \leq h-1$, h — the class-number of K . As the only partition of R_K which consists of one set is $A_1 = R_K$, the theorem of Carlitz [3] in this terminology will have the form:

$$l(K) = 1 \text{ for } h > 1 \text{ if and only if } h = 2.$$

THEOREM 2. $l(K) = 2$ if and only if $h = 3, 4, 6$.

THEOREM 3. $\lim_{h \rightarrow \infty} l(K) = \infty$.

3. Let $G_m(x)$ be the number of non-associated integers of K with norms not exceeding x in absolute value and whose factorizations into irreducibles in K have exactly m distinct lengths, and similarly let $G'_m(x)$ be the number of natural numbers not exceeding x whose factorizations into irreducibles in K have exactly m distinct lengths (see [2—4]). (For $m = 1$ $G_1(x)$ and $G'_1(x)$ are the counting functions of $G(K)$ and $G'(K)$).

THEOREM 4. If K/Q is finite then

$$G_m(x) = (C + o(1)) x (\log x)^{-1 + \frac{t(K)}{h}} (\log \log x)^d,$$

where $t(K) = \max \{|U| : U \subset H(K), U \in (*)\}$, A is a positive constant depending only on $H(K)$ and C is positive and depends on K .

THEOREM 5. If K/Q is finite then

$$G'_m(x) = (C + O(1)) x (\log x)^{-b} (\log \log x)^B,$$

where C and B are positive and b is some nonnegative constant depending only on $H(K)$.

The proofs will be published elsewhere.

INSTITUTE OF MATHEMATICS, UNIVERSITY, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW
(INSTYTUT MATEMATYKI, UNIWERSYTET WROCLAWSKI)

REFERENCES

- [1] L. Carlitz, *A characterization of algebraic number fields with class number two*, Proc. Amer. Math. Soc., **11** (1960), 301—392.
[2] W. Narkiewicz, *On algebraic number fields with nonunique factorization*, Coll. Math., **12** (1964), 59—68.
[3] ———, *On algebraic number fields with nonunique factorization II*, *ibid.*, **15** (1966), 49—58.
[4] ———, *On natural numbers having unique factorization in a quadratic number field*, Acta Arith., **12** (1966), 1—22.

Я. Слива, О длине разложений в числовых полях

Содержание. В работе приведены некоторые обобщения теоремы Карлица о длине разложений элементов в числовых полях с числом классов два. Поданы тоже асимптотические формулы для количества целых чисел нормального поля K у которых количество длин разложений на неприводимые сомножители постоянно и норма которых меньше x ($x > 0$).