

## A Note on Factorizations in Algebraic Number Fields

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**Summary.** We give an asymptotic formula for the number of natural numbers  $\leq x$  which have at most  $m$  ( $m$  a natural number) distinct factorizations in a given number field.

1. Let  $K$  be an algebraic number field. Denote by  $F_m(K)$  the set of all natural numbers which have at most  $m$  distinct factorizations into irreducibles in  $K$  and by  $F_m(x)$  the number of integers in  $F_m(K)$  which does not exceed  $x$ .

The asymptotic formula for  $F_1(x)$  was obtained in [2, 3] (in [3] even with the remainder term). Allen in [4] gives an upper bound for  $F_m(x)$ ,  $m \geq 1$ .

We use the method of [1] where a similar problem for algebraic integers in  $K$  was solved, and prove

**THEOREM.** *If  $K$  is an algebraic number field with class number  $h \neq 1$ , then*

$$F_m(x) = (C + o(1)) \frac{x (\log \log x)^{N_m}}{(\log)^{1-q}},$$

where  $q$  is the density of primes which have only principal ideals in their decompositions into prime ideals and  $N_m$  is a positive constant depending on  $K$  and  $m$ .

2. Let  $p$  be a rational prime and

$$(p) = \mathfrak{P}_1 \cdot \dots \cdot \mathfrak{P}_g$$

its decomposition into prime ideals. The collection  $\{X_1, \dots, X_g\}$  of elements of class-group  $H(K)$ , such that  $\mathfrak{P}_i \in X_i$ , will be called an orbit of  $p$ . If  $O$  is such an orbit then by  $P_O$  we denote the set of all rational primes which have  $O$  as their orbit. The following lemma is a consequence of the proof of a similar result, obtained by Odoni [3].

**LEMMA 1.** *For any orbit  $O$ , the set  $P_O$  is finite or*

$$\sum_{p \in P_O} p^{-s} = q(O) \log \frac{1}{s-1} + g_0(s),$$

where  $q(O) > 0$  and  $g_0(s)$  is regular for  $\text{Re } s \geq 1$ .

Denote by  $P'$  the set of all primes which have in their orbits only unit elements of  $H(K)$ , and let  $O_1, \dots, O_t$  denote all orbits  $\neq \{E, \dots, E\}$ ,  $E$  — a unit element of  $H(K)$ . The set of primes corresponding to the orbits  $O_1, \dots, O_t$  will be denoted by  $P_1, \dots, P_t$ , respectively. Assume that  $P_1, \dots, P_r$  ( $r \leq t$ ) have positive densities, while all the others are finite.

If  $n$  is a natural number

$$n = n' p_{1,1}^{a_1^{(1)}} \dots p_{1,k_1}^{a_{k_1}^{(1)}} \dots p_{t,1}^{a_t^{(t)}} \dots p_{t,k_t}^{a_{k_t}^{(t)}},$$

where  $n'$  has only prime divisors from  $P'$  and  $p_{i,j} \in P_i$  ( $1 \leq i \leq t, 1 \leq j \leq k_i$ ), then we shall say that the system

$$\tau(n) = \langle \{a_1^{(1)}, \dots, a_{k_1}^{(1)}\}, \dots, \{a_1^{(t)}, \dots, a_{k_t}^{(t)}\} \rangle$$

is the type of  $n$ . Define

$$d(\tau) = \mathcal{N} \{a_i^{(j)} = 1 : 1 \leq j \leq r, 1 \leq i \leq k\}.$$

LEMMA 2. Let  $A$  be any set of natural numbers satisfying the following conditions:

- (i) If  $n \in A$ ,  $\tau(m) = \tau(n)$  then  $m \in A$ .
- (ii) There is a constant  $B$  such that  $n \in A$  implies  $d(\tau(n)) \leq B$ .

Then for the number  $A(x)$  of numbers  $n$  in  $A$  with  $n \leq x$  one has

$$A(x) = (C + O(1)) \frac{x (\log \log x)^{N_A}}{(\log x)^{1-q}},$$

where  $q$  is the density of  $P'$ ,  $N_A = \max \{d(\tau(n)) : n \in A\} \leq B$  (see [1], 3 (b)).

Proof of the theorem. It is sufficient now to show that the set  $F_m(K)$  satisfies conditions (i), (ii) of Lemma 2. If  $\tau(n) = \tau(m)$  then  $m$  and  $n$  have the same type in the sense of [1], and so they have the same number of factorizations in  $K$ . This proves (i). Condition (ii) is a consequence of Lemma 1 of [1].

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Я. Слива, О разложениях целых рациональных чисел в числовых полях

Содержание. В работе доказана асимптотическая формула для количества положительных целых чисел  $\leq x$ , у которых не больше чем  $m$  различных разложений на неприводимые множители в данном числовом поле.